Week 1: Why Learning Theory?

The No-Free-Lunch Theorem

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DSC 190/291 Topics: Learning Theory

Today

Course Overview

Logistics

Week 1: Online Prediction

The No-Free-Lunch Theorem

How Structure Helps

Course Overview

What is this course about?

Machine Learning is Everywhere

Examples:

- Image classification
- Speech recognition
- Large language models (GPT, Claude)
- Recommendations (Netflix, Spotify)
- Medical diagnosis
- Fraud detection

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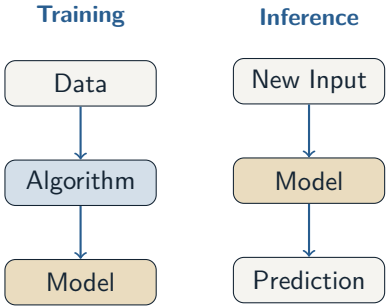
All involve learning a **prediction rule** from data.



The Central Question

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But first: **Is learning always possible?**

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There is no free lunch for learning.

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The resolution:

- Real problems have **structure**

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Learning Theory: Rigorous study of what can be learned, how much data is needed, and how efficiently.

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- Create human-readable explanations alongside

Paper Proof vs Formal Proof

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(Requires induction—why?)

Why the Difference?

Lean's addition is defined by recursion on the *second* argument:

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Formal proof exposes details that paper proofs gloss over.

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- AI tools assist proof generation
- Lower barrier to entry than ever before

AI + Formal Proof: Rapid Progress

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- Exposes hidden assumptions
- Clarifies dependencies between lemmas
- Catches subtle errors in published proofs

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New challenge: How do we make formal proofs accessible to humans?

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A proof is not finished when it compiles.

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- Proof blueprints (lemma dependencies)
- Visualizations and trace tables

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1. Learning Theory

- Definitions
- Theorems
- Calculations

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AI assistance opens new possibilities for proof and presentation.

Logistics

How this course runs

Weekly Schedule

Wednesdays, 5:00 – 7:50 PM

Lecture		Lecture		Workshop
5:00 – 5:50	break	6:00 – 6:50	break	7:00 – 7:50

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Workshop: Student presentations and feedback

Quarter-Long Project

Milestones:

1. Choose a learning theory result to formalize

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Weekly assignments: Theory problems + brief project progress report

Project Output

1. **Formalized results** — compiling Lean proofs

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3. **AI workflow** — `CLAUDE.md`, `AGENTS.md`, skills, and other AI artifacts you develop along the way

Project Repository

Each team sets up a Git repo to track their work.

Repo should contain:

- Lean project files
- Presentation materials
- AI workflow artifacts (`CLAUDE.md`, `AGENTS.md`, skills, etc.)
- README

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I will maintain a **course repo** with a list of all project repos.

Grading

Component	Weight
Weekly Assignments	30%
Quarter-Long Project	40%
Participation	30%

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Participation includes:

- Attending and engaging in lecture
- Workshop presentations
- Giving feedback to peers

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AI is a collaborator, not an oracle.

Week 1's Topic

The first model of the course

The Basic Prediction Setup

Instance space $\mathcal{X}$: set of possible inputs

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Running example: Threshold classifier on $\mathbb{R}$

$$h_\theta(x) = \mathbf{1}[x \geq \theta]$$

- $h_5(3) = 0$ (below threshold)
- $h_5(7) = 1$ (above threshold)
- $h_5(5) = 1$ (at threshold)

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Prediction comes before label is revealed.

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Examples:

- **Constant:** $h_t(x) = 1$ for all x
- **Memorize:** $h_t(x) =$ last label seen for x, else default

Counting Mistakes

A **mistake** occurs when $\hat{y}_t \neq y_t$.

Total mistakes: $M_T = \sum_{t=1}^{T} \mathbf{1}[\hat{y}_t \neq y_t]$

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This is exactly the question we address next.

No-Free-Lunch

Why learning needs assumptions

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If no: we need to understand **what assumptions** make learning possible.

Theorem

For any finite $\mathcal{X}$ and any deterministic learner A, there exists:

- a function $f\colon \mathcal{X} \to \{0,1\}$, and
- a sequence $x_1, \ldots, x_n$ with $n = |\mathcal{X}|$ (all distinct)

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In words: No learner can guarantee fewer than $|\mathcal{X}|$ mistakes in the worst case.

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Why is f well-defined? Each x_t is new — no previous value is overwritten.

Why n mistakes? By construction, $\hat{y}_t \neq f(x_t)$ on every round.

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What if we have **prior knowledge** about the labeling rule?

How Structure Helps

When assumptions rescue learning

Structure Helps: Threshold Example

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2. For point x: predict $\hat{y} = \mathbf{1}[x > \frac{a+b}{2}]$

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Structure Helps: Threshold Example

Setup: $\mathcal{X} = [0, 1]$, predict labels $y \in \{0, 1\}$ for points $x \in \mathcal{X}$

Assumption: There exists $\theta^* \in [0, 1]$ such that $y = \mathbf{1}[x > \theta^*]$

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At most $\lceil \log_2(1/\varepsilon) \rceil$ mistakes for precision ε

Hypothesis Class = Prior Knowledge

A **hypothesis class** $\mathcal{H} \subseteq \mathcal{Y}^{\mathcal{X}}$ is a set of candidate labeling functions.

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Interpretation: $\mathcal{H}$ encodes our *prior knowledge* about the problem

- Smaller $\mathcal{H}$ = stronger assumption = easier to learn
- Larger $\mathcal{H}$ = weaker assumption = harder to learn

Generalizing: Finite Hypothesis Classes

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Question: How many mistakes are needed to identify h^*?

The Halving Algorithm

Idea: Maintain *version space* $V_t \subseteq \mathcal{H}$ = hypotheses consistent with history

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Key observation: True h^* is never eliminated (always in V_t)

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Claim: Each mistake eliminates at least half of V_t

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- Start with $|V_1| = |\mathcal{H}|$
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Halving makes at most $\lfloor \log_2 |\mathcal{H}| \rfloor$ mistakes

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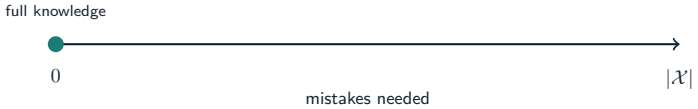
Key insight: Need a notion of "complexity" beyond cardinality

Coming up: VC dimension, Littlestone dimension, ...

The Tradeoff: Expressiveness vs. Mistakes

$\mathcal{X} = 2^D$ (emails as subsets of dictionary D), $\mathcal{Y} = \{0, 1\}$ (spam/not spam)
keyword $=$ "free" $\in x$; conjunction $=$ "free" $\wedge$ "click" $\wedge$ "win" $\in x$



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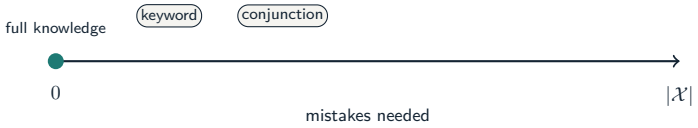

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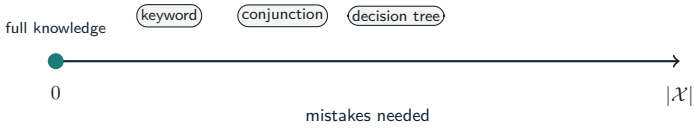

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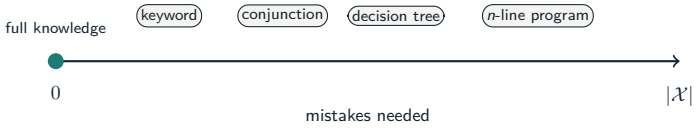

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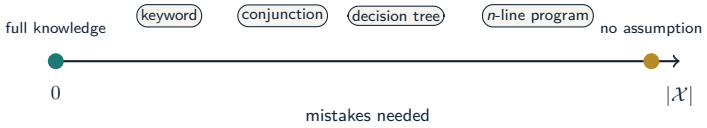

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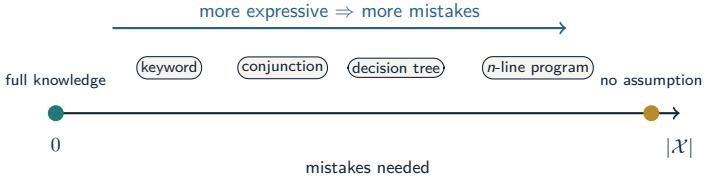

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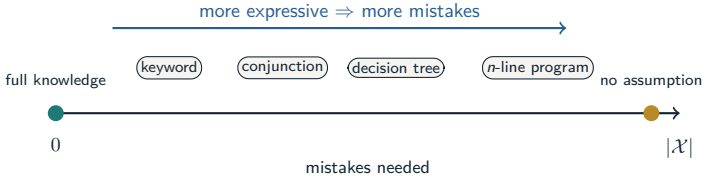

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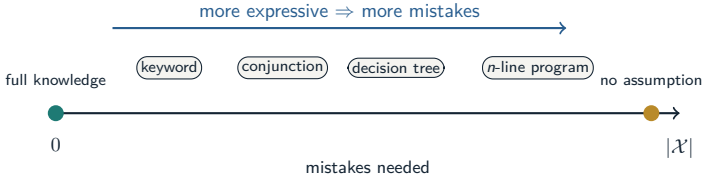

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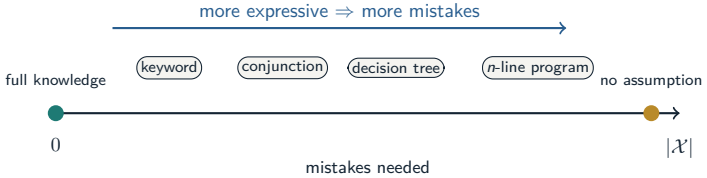

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We want hypothesis classes that are:

1. Expressive (capture interesting concepts with low $\log |\mathcal{H}|$)
2. Computationally efficiently learnable

An Important Class: Linear Predictors

Setup: Choose features $\phi_1(x), \phi_2(x), \ldots, \phi_d(x)$

$$\phi : \mathcal{X} \to \mathbb{R}^d, \quad \phi(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_d(x))$$

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Interpretation:

- The feature map $\phi(x)$ encodes our prior knowledge
- The linear separator w is what we learn

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Problem: $|\mathcal{H}| = \infty$ (uncountably many choices of w, θ)

The Challenge: Infinite Hypothesis Class

Example: Threshold functions in 1D

$$\mathcal{H} = \left\{ h_\theta(x) = \mathbf{1}[x \leq \theta] \;\middle|\; \theta \in \mathbb{R} \right\}$$

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- $x_{t+1} = x_t + y_t \cdot 2^{-(t+1)}$
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Key observation: Adversary exploits *infinite precision* — points get arbitrarily close to the threshold.

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Question: Can we avoid both infinite mistakes AND exponential runtime?

So We Can't Learn Linear Predictors?

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The key: Margin γ is the "right" complexity measure for linear predictors

The Perceptron Algorithm (Rosenblatt 1958)

Algorithm:

1. Initialize $w_1 = 0$
2. At round t:
   - Receive x_t, predict $\hat{y}_t = \text{sign}(\langle w_t, \phi(x_t) \rangle)$
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No need to enumerate hypotheses — just update weights!

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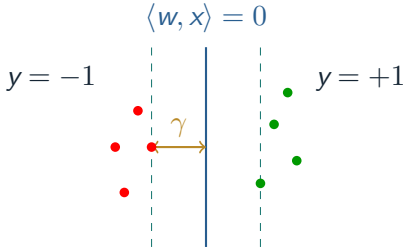

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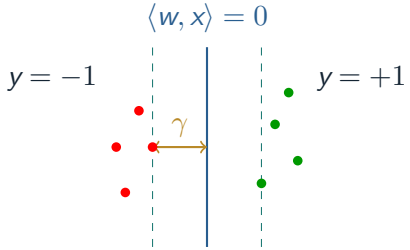

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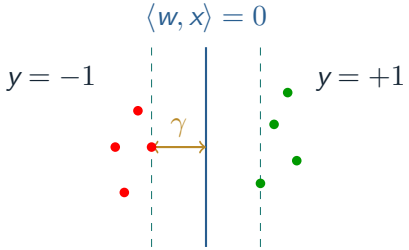

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The threshold counterexample has $\gamma \to 0$ — adversary pushes points arbitrarily close to boundary!

Perceptron: Mistake Bound

Theorem (Novikoff 1962): If data is linearly separable with margin γ, and $\|\phi(x_t)\| \leq R$ for all t, then Perceptron makes at most

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A problem in $d = 1{,}000{,}000$ dimensions with large margin is easier than a problem in $d = 2$ with tiny margin.

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Recall: $w_1 = 0$; predict $\hat{y}_t = \text{sign}(\langle w_t, \phi(x_t)\rangle)$; on mistake, $w_{t+1} \leftarrow w_t + y_t\phi(x_t)$.
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The margin assumption gives $y_i\langle w^*, \phi(x_i)\rangle \ge \gamma$ for every i, so:

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Telescoping from $\|w_1\|^2 = 0$: each mistake adds at most R^2, so $\|w_{t+1}\|^2 \leq R^2 M_t$.

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Dividing both sides by $\sqrt{M_t}$:

$$\gamma\sqrt{M_t} \leq R \quad \Rightarrow \quad M_t \leq \frac{R^2}{\gamma^2}$$

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4. **Efficient algorithms exist**
   Perceptron: $O(d)$ per round vs Halving: $O(|\mathcal{H}|)$ per round

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This motivates the statistical learning model — next week.

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Concepts introduced:

- Instance space, label space, hypothesis
- Online prediction model, mistake counting
- No-Free-Lunch theorem