# Week 1: Why Learning Theory?

## The No-Free-Lunch Theorem

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DSC 190/291 Topics: Learning Theory

# Today

# Course Overview

What is this course about?

## Machine Learning is Everywhere

**Examples:**

- Image classification
- Speech recognition
- Large language models (GPT, Claude)
- Recommendations (Netflix, Spotify)
- Medical diagnosis
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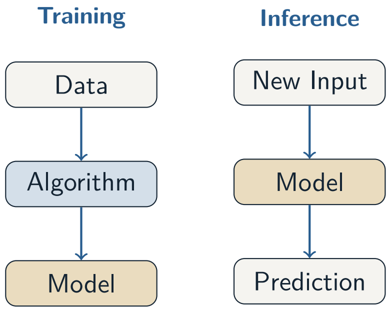
All involve learning a **prediction rule** from data.



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But first: **Is learning always possible?**

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**There is no free lunch for learning.**

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**Learning Theory:** Rigorous study of what can be learned, how much data is needed, and how efficiently.

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- Create human-readable explanations alongside

# Paper Proof vs Formal Proof

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(Requires induction—why?)

## Why the Difference?

Lean's addition is defined by recursion on the *second* argument:

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**Formal proof exposes details that paper proofs gloss over.**

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**Formal proof forces explicit assumptions and dependencies.**

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- AI tools assist proof generation
- Lower barrier to entry than ever before

# AI + Formal Proof: Rapid Progress

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- Exposes hidden assumptions
- Clarifies dependencies between lemmas
- Catches subtle errors in published proofs

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**New challenge:** How do we make formal proofs accessible to humans?

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**A proof is not finished when it compiles.**

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- Proof blueprints (lemma dependencies)
- Visualizations and trace tables

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**Complete artifact = Verified proof + Clear explanation**

*AI assistance opens new possibilities for proof and presentation.*

# Logistics

How this course runs

## Weekly Schedule

**Wednesdays, 5:00 – 7:50 PM**

| Lecture | | Lecture | | Workshop |
|---|---|---|---|---|
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**Workshop:** Student presentations and feedback

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**Weekly assignments:** Theory problems + brief project progress report

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# Project Repository

Each team sets up a Git repo to track their work.

**Repo should contain:**

- Lean project files
- Presentation materials
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I will maintain a **course repo** with a list of all project repos.

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| Component | Weight |
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| Weekly Assignments | 30% |
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**Participation includes:**

- Attending and engaging in lecture
- Workshop presentations
- Giving feedback to peers

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**AI is a collaborator, not an oracle.**

# Week 1's Topic

The first model of the course

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$$h_\theta(x) = \mathbf{1}[x \geq \theta]$$

- $h_5(3) = 0$ (below threshold)
- $h_5(7) = 1$ (above threshold)
- $h_5(5) = 1$ (at threshold)

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**Prediction comes before label is revealed.**

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**Examples:**

- **Constant:** $h_t(x) = 1$ for all $x$
- **Memorize:** $h_t(x) =$ last label seen for $x$, else default

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A **mistake** occurs when $\hat{y}_t \neq y_t$.

Total mistakes: $M_T = \sum_{t=1}^{T} \mathbf{1}[\hat{y}_t \neq y_t]$

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This is exactly the question we address next.

# No-Free-Lunch

Why learning needs assumptions

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If no: we need to understand **what assumptions** make learning possible.

# No-Free-Lunch Theorem

**Theorem**

For any finite $\mathcal{X}$ and any deterministic learner $A$, there exists:

- a function $f \colon \mathcal{X} \to \{0, 1\}$, and
- a sequence $x_1, \ldots, x_n$ with $n = |\mathcal{X}|$ (all distinct)

such that $A$ makes $n$ mistakes on the sequence $(x_t, y_t = f(x_t))_{t=1}^{n}$.

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**In words:** No learner can guarantee fewer than $|\mathcal{X}|$ mistakes in the worst case.

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**Why $n$ mistakes?** By construction, $\hat{y}_t \neq f(x_t)$ on every round.

# The Lesson from NFL

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**The way out:** Assume the labeling rule belongs to a known hypothesis class $\mathcal{H}$.

What if we have **prior knowledge** about the labeling rule?

# How Structure Helps

When assumptions rescue learning

# Structure Helps: Threshold Example

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**At most $\lceil \log_2(1/\varepsilon) \rceil$ mistakes for precision $\varepsilon$**

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- Learner knows $\mathcal{H}$, but not which $h^* \in \mathcal{H}$

**Interpretation:** $\mathcal{H}$ encodes our *prior knowledge* about the problem

- Smaller $\mathcal{H}$ = stronger assumption = easier to learn
- Larger $\mathcal{H}$ = weaker assumption = harder to learn

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**Question:** How many mistakes are needed to identify $h^*$?

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**Key observation:** True $h^*$ is never eliminated (always in $V_t$)

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- Start with $|V_1| = |\mathcal{H}|$
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**Halving makes at most $\lfloor \log_2 |\mathcal{H}| \rfloor$ mistakes**

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**Key insight:** Need a notion of "complexity" beyond cardinality

**Coming up: VC dimension, Littlestone dimension, ...**

# The Tradeoff: Expressiveness vs. Mistakes

$\mathcal{X} = 2^D$ (emails as subsets of dictionary $D$), $\mathcal{Y} = \{0, 1\}$ (spam/not spam)

keyword = "free" $\in x$;    conjunction = "free" $\wedge$ "click" $\wedge$ "win" $\in x$


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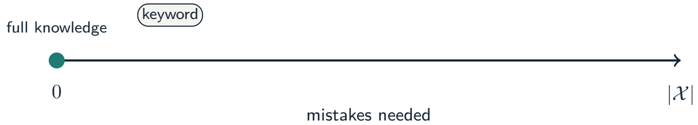


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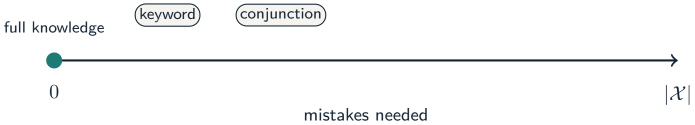


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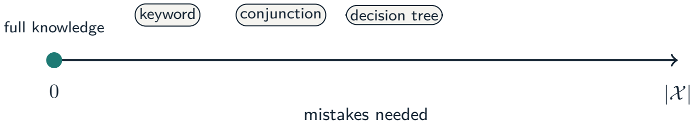


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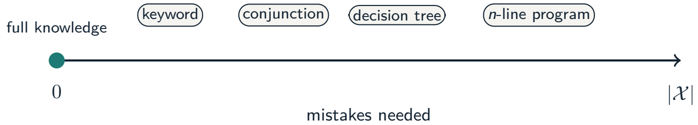


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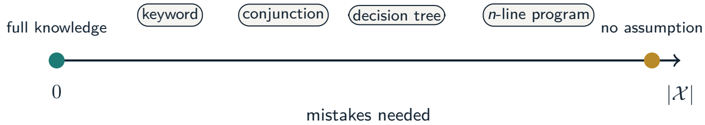


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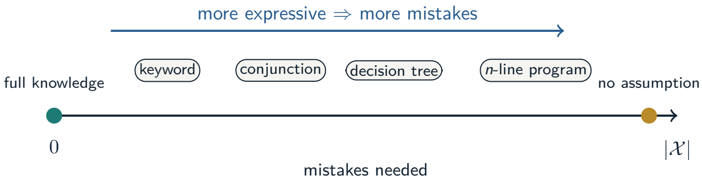

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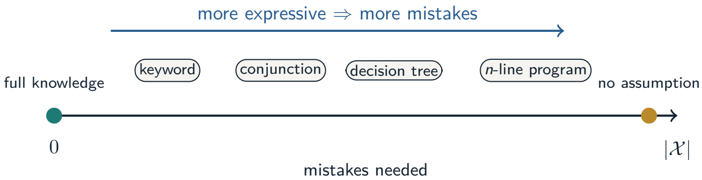


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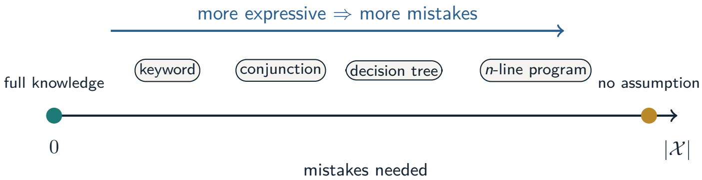


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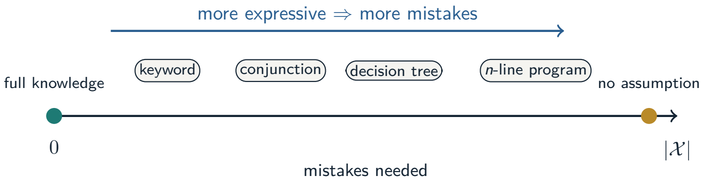


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**We want hypothesis classes that are:**

1. Expressive (capture interesting concepts with low $\log |\mathcal{H}|$)
2. Computationally efficiently learnable

# An Important Class: Linear Predictors

**Setup:** Choose features $\phi_1(x), \phi_2(x), \ldots, \phi_d(x)$

$$\phi : \mathcal{X} \to \mathbb{R}^d, \quad \phi(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_d(x))$$

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**Interpretation:**

- The feature map $\phi(x)$ encodes our prior knowledge
- The linear separator $w$ is what we learn

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**Problem:** $|\mathcal{H}| = \infty$ (uncountably many choices of $w, \theta$)

# The Challenge: Infinite Hypothesis Class

**Example:** Threshold functions in 1D

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**Key observation:** Adversary exploits *infinite precision* — points get arbitrarily close to the threshold.

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# From Infinite to Finite: Discretization

**Idea:** Make $\mathcal{H}$ finite by limiting precision

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**Question:** Can we avoid both infinite mistakes AND exponential runtime?

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**The key:** Margin $\gamma$ is the "right" complexity measure for linear predictors

# The Perceptron Algorithm (Rosenblatt 1958)

**Algorithm:**

1. Initialize $w_1 = 0$
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**No need to enumerate hypotheses — just update weights!**

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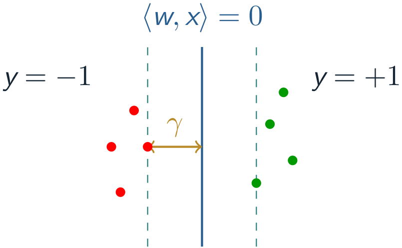

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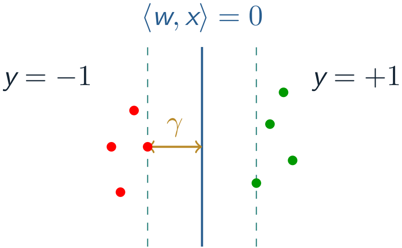


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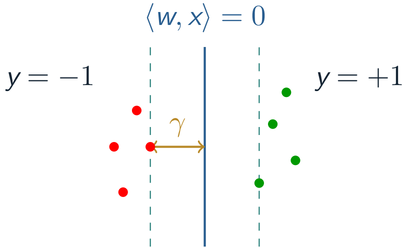


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**The threshold counterexample has $\gamma \to 0$ — adversary pushes points arbitrarily close to boundary!**

## Perceptron: Mistake Bound

**Theorem (Novikoff 1962):** If data is linearly separable with margin $\gamma$, and $\|\phi(x_t)\| \leq R$ for all $t$, then Perceptron makes at most

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A problem in $d = 1{,}000{,}000$ dimensions with large margin is easier than a problem in $d = 2$ with tiny margin.

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**Recall:** $w_1 = 0$; predict $\hat{y}_t = \text{sign}(\langle w_t, \phi(x_t)\rangle)$; on mistake, $w_{t+1} \leftarrow w_t + y_t\phi(x_t)$.
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The margin assumption gives $y_i\langle w^*, \phi(x_i)\rangle \geq \gamma$ for every $i$, so:

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Telescoping from $\|w_1\|^2 = 0$: each mistake adds at most $R^2$, so $\|w_{t+1}\|^2 \leq R^2 M_t$.

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Dividing both sides by $\sqrt{M_t}$:

$$\gamma\sqrt{M_t} \leq R \quad \Rightarrow \quad M_t \leq \frac{R^2}{\gamma^2}$$

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4. **Efficient algorithms exist**
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**This motivates the statistical learning model — next week.**

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**Concepts introduced:**

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- Online prediction model, mistake counting
- No-Free-Lunch theorem